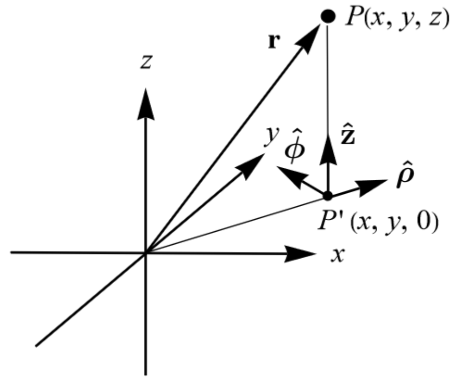
Problem 1.48

Find expressions for the unit vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$ of cylindrical polar coordinates (Problem 1.47) in terms of the Cartesian $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. Differentiate these expressions with respect to time to find $d\hat{\boldsymbol{\rho}}/dt$, $d\hat{\boldsymbol{\phi}}/dt$, and $d\hat{\mathbf{z}}/dt$.

Solution

The unit vectors in cylindrical coordinates (ρ, ϕ, z) are illustrated below.



$\hat{\boldsymbol{\rho}}$ points radially outward from the z-axis; $\hat{\boldsymbol{\phi}}$ is perpendicular to both $\hat{\boldsymbol{\rho}}$ and $\hat{\mathbf{z}}$, pointing in the direction of increasing ϕ; and $\hat{\mathbf{z}}$ points in the direction of the z-axis.

$$\begin{aligned}
\hat{\boldsymbol{\rho}} &= \frac{\boldsymbol{\rho}}{|\boldsymbol{\rho}|} \\
&= \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + 0^2}} \\
&= \frac{x}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{x}} + \frac{y}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}} \\
&= \frac{x}{\rho}\,\hat{\mathbf{x}} + \frac{y}{\rho}\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}} \\
&= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{\phi}} &= \hat{\mathbf{z}} \times \hat{\boldsymbol{\rho}} \\
&= \hat{\mathbf{z}} \times (\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) \\
&= \cos\phi\,(\hat{\mathbf{z}} \times \hat{\mathbf{x}}) + \sin\phi\,(\hat{\mathbf{z}} \times \hat{\mathbf{y}}) + 0\,(\hat{\mathbf{z}} \times \hat{\mathbf{z}}) \\
&= \cos\phi\,(\hat{\mathbf{y}}) + \sin\phi\,(-\hat{\mathbf{x}}) + 0\,(\mathbf{0}) \\
&= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}
\end{aligned}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

Take the first derivative of each cylindrical unit vector with respect to t, noting that the derivative of each Cartesian unit vector with respect to time is zero.

$$\begin{aligned}
\frac{d\hat{\boldsymbol{\rho}}}{dt} &= \frac{d}{dt}(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) \\
&= \frac{d}{dt}(\cos\phi\,\hat{\mathbf{x}}) + \frac{d}{dt}(\sin\phi\,\hat{\mathbf{y}}) \\
&= \frac{d}{dt}(\cos\phi)\,\hat{\mathbf{x}} + \cos\phi\,\frac{d\hat{\mathbf{x}}}{dt} + \frac{d}{dt}(\sin\phi)\,\hat{\mathbf{y}} + \sin\phi\,\frac{d\hat{\mathbf{y}}}{dt} \\
&= \left(-\sin\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{x}} + \cos\phi\,\underbrace{\frac{d\hat{\mathbf{x}}}{dt}}_{=\mathbf{0}} + \left(\cos\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{y}} + \sin\phi\,\underbrace{\frac{d\hat{\mathbf{y}}}{dt}}_{=\mathbf{0}} \\
&= \left(-\sin\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{x}} + \left(\cos\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{y}} \\
&= \frac{d\phi}{dt}(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) \\
&= \frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\hat{\boldsymbol{\phi}}}{dt} &= \frac{d}{dt}(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) \\
&= \frac{d}{dt}(-\sin\phi\,\hat{\mathbf{x}}) + \frac{d}{dt}(\cos\phi\,\hat{\mathbf{y}}) \\
&= \frac{d}{dt}(-\sin\phi)\,\hat{\mathbf{x}} - \sin\phi\,\frac{d\hat{\mathbf{x}}}{dt} + \frac{d}{dt}(\cos\phi)\,\hat{\mathbf{y}} + \cos\phi\,\frac{d\hat{\mathbf{y}}}{dt} \\
&= \left(-\cos\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{x}} - \sin\phi\,\underbrace{\frac{d\hat{\mathbf{x}}}{dt}}_{=\mathbf{0}} + \left(-\sin\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{y}} + \cos\phi\,\underbrace{\frac{d\hat{\mathbf{y}}}{dt}}_{=\mathbf{0}} \\
&= \left(-\cos\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{x}} + \left(-\sin\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{y}} \\
&= -\frac{d\phi}{dt}(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) \\
&= -\frac{d\phi}{dt}\,\hat{\boldsymbol{\rho}}
\end{aligned}$$

$$\frac{d\hat{\mathbf{z}}}{dt} = \mathbf{0}$$

Therefore,

$$\frac{d\hat{\boldsymbol{\rho}}}{dt} = \frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} \qquad \text{and} \qquad \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\frac{d\phi}{dt}\,\hat{\boldsymbol{\rho}} \qquad \text{and} \qquad \frac{d\hat{\mathbf{z}}}{dt} = \mathbf{0}.$$